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A simultaneous process for convergence acceleration and error control

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Abstract: Let (x_n) be a real sequence, converging to the limit x^* and such that $\Delta x_n = x_{n+1} - x_n = \lambda^n n^\nu \Sigma_{i \geqslant 0} a_i / n^i$ $(a_0 \neq 0, \lambda \neq 0)$ for n large enough.

Here we propose a repeated process, using two transformations (namely Aitken's Δ^2 process and Brezinski's θ_2 algorithm) which provide a sequence of intervals asymptotically containing x^*.

In particular, we introduce, for $\lambda = 1$, a version of the modified iterated Δ^2 process, based on the convergence orders, so generalizing a method proposed by Sablonnière. We establish that the order ν_p, used in the pth iteration, is $\nu_{p-1} - k_{p-1} + 1$, $k_{p-1} \geqslant 3$ being an integer. We give an estimation method for ν_p confirming the choice of k_{p-1}.

Keywords: Asymptotic expansion, convergence orders, convergence acceleration, error bounds.

1. Introduction

In this paper, we propose a process for controlling the error in convergence acceleration methods. In practice, it is not enough to know that a given transformation accelerates the convergence of a family of sequences. It is also important to have an estimation of the error, thus leading to a stopping criterion. Here we give a process for estimating the limit with two accelerative transformations, inspired by that of Stirling and Andoyer [14].

The method is a natural consequence of the generalization studied in [2, Chapter IV] of the error control process of Brezinski [6].

The proofs of our propositions are based upon elementary properties of power series and analytic functions. We use techniques similar to those used in [11] by Lubkin.

Numerical examples illustrate the results of this paper.

2. Preliminary results

Let $C_{\mathbb{R}}$ denote the set of convergent real sequences. In the following, $x^*, y^*, \ldots$ will refer to the limits of the sequences $(x_n), (y_n), \ldots$ of $C_{\mathbb{R}}$. Let us first recall some definitions.

Definition 2.1. Let $(s_n) \in C_{\mathbb{R}}$. We assume that

$$\lim_{n \to +\infty} \frac{s_{n+1} - s^*}{s_n - s^*} = \mu \in \mathbb{R}^*.$$

If $-1 \leqslant \mu < 1$, (s_n) is said to be *linearly convergent*: $(s_n) \in \mathrm{LIN}$. If $\mu = 1$, it is said to be *logarithmically convergent*, or simpler it is *logarithmic*: $(s_n) \in \mathrm{LOG}$.

Definition 2.2. We denote by F the family of sequences of $C_{\mathbb{R}}$ such that the differences of two successive terms, $\Delta s_n = s_{n+1} - s_n$ admits an asymptotic expansion (A.E.) in $1/n$ of the form

$$\Delta s_n = \lambda^n n^\nu \left(a_0 + \frac{a_1}{n} + \frac{a_2}{n^2} + \cdots \right) \tag{1}$$

for n large enough ($a_0 \neq 0$, $\lambda \neq 0$, $\nu \in \mathbb{R}$).

Let us first recall an important result, due to Wimp [15], on which the following others are based.

Lemma 2.3. *Let* $(s_n) \in C_{\mathbb{R}}$ *and having the A.E.* (1). *Then, for* $n \to \infty$,

$$s^* - s_{n+1} = \begin{cases} \dfrac{\lambda^{n+1}}{1-\lambda} n^\nu \left(a_0 + \dfrac{b_1}{n} + \dfrac{b_2}{n^2} + \cdots \right), & \text{if } -1 \leqslant \lambda < 1 \ (\lambda \neq 0), \\ -\dfrac{n^{\nu+1}}{\nu+1} \left(a_0 + \dfrac{c_1}{n} + \dfrac{c_2}{n^2} + \cdots \right), & \text{if } \lambda = 1 \text{ and } \nu < -1. \end{cases}$$

We can see that, in the first case, (s_n) is a linearly convergent sequence and we shall write $(s_n) \in F_{\mathrm{LIN}}$. In the second case, it is a logarithmic sequence and we write $(s_n) \in F_{\mathrm{LOG}}$.

Note that the condition $\nu < -1$ guarantees the convergence of the sequence when $\lambda = 1$.

Furthermore, if $(s_n) \in F_{\mathrm{LIN}}$ or F_{LOG}, there is always an index N such that $\forall n \geqslant N$, $\Delta s_n \neq 0$ and the ratio $\Delta s_{n+1}/\Delta s_n$ exists. This leads us to the following result.

Lemma 2.4. *If* $(s_n) \in F$, *then, for* n *large enough,* Δs_n *and* $s^* - s_n$ *have the same sign.*

Proof. (i) If $(s_n) \in F_{\mathrm{LIN}}$, then we have:

$$\Delta s_n \sim \lambda^n n^\nu a_0 \quad \text{and} \quad s^* - s_n \sim \frac{\lambda^n n^\nu a_0}{1-\lambda} \qquad (n \to \infty).$$

Thus $\lim_{n \to +\infty} \Delta s_n/(s^* - s_n) = 1 - \lambda > 0$ because $-1 \leqslant \lambda < 1$.

(ii) If $(s_n) \in F_{\mathrm{LOG}}$, then we have:

$$\Delta s_n \sim n^\nu a_0 \quad \text{and} \quad s^* - s_{n+1} \sim -\frac{n^{\nu+1}}{\nu+1} a_0 \qquad (n \to \infty).$$

Thus $\lim_{n \to +\infty} \Delta s_{n+1}/(s^* - s_{n+1}) = -(\nu+1) > 0$ because $\nu < -1$. $\square$

Remark 2.5. Throughout this work, we will deal with operations with asymptotic expansions. They are thoroughly justified owing to the link with analytic functions (Ritt's theorem). For more details, the reader should consult [1,9,11,13]. In particular, if $\phi_n = a_0 + a_1/n + a_2/n^2 + \cdots$

exists for $n \geqslant N > 0$ and if we set $\phi(t) = a_0 + a_1 t + a_2 t^2 + \cdots$, then ϕ is defined for $t = 1/N$, and then for $|t| \leqslant 1/N$. Consequently, ϕ defines an analytic function in this disc and we have: $\phi(1/n) = \phi_n$ $\forall n \geqslant N$.

If we set $\phi_+(t) = \phi(t/(1+t))$ and $\phi_-(t) = \phi(t/(1-t))$, then

$$\phi_{n+1} = \phi_+(1/n) \quad \text{and} \quad \phi_{n-1} = \phi_-(1/n) \qquad (n > 1).$$

Thus, we have $\phi_{n+1} = \sum_{i \geqslant 0} b_i/n^i$ with $b_0 = a_0$, $b_1 = a_1$ and $b_j = b_j(1)$ where

$$b_j(m) = \sum_{i=0}^{j-1} (-1)^{j-1-i} \binom{j-1}{i} a_{i+m} = \Delta^{j-1} a_m = \Delta\left(\Delta^{j-2} a_m\right).$$

The symbol $\binom{k}{i}$ is defined for arbitrary integers i, $k \geqslant i$ by

$$\binom{k}{0} = 1, \qquad \binom{k}{i} = \frac{k(k-1) \cdot \ldots \cdot (k-i+1)}{1 \cdot 2 \cdot \ldots \cdot i}, \quad i \geqslant 1.$$

3. Error bounds for F_{LIN}

In this section, we shall consider the family F_{LIN}.

Lemma 3.1. *If* $(s_n) \in F_{\mathrm{LIN}}$, *then*:

(i) $$\lambda_n = \frac{\Delta s_{n+1}}{\Delta s_n} = \alpha_0 + \frac{\alpha_1}{n} + \frac{\alpha_2}{n^2} + \cdots \quad \textit{for } n \to +\infty, \textit{ with } \alpha_0 = \lambda,$$

(ii) $$n^{k+1}\, \Delta\lambda_n \sim -k\alpha_k \quad \textit{and} \quad n^{k+2}\, \Delta^2 \lambda_n \sim k(k+1)\alpha_k, \qquad \textit{for } n \to +\infty,$$

where k is the smallest positive integer such that $\alpha_i \neq 0$.

Proof.

$$\exists N > 0\ \forall n \geqslant N\ \Delta s_n = n^\nu \lambda^n \sum_{i \geqslant 0} \frac{a_i}{n^i} = n^\nu \lambda^n \phi_n.$$

From Remark 2.5, we can associate with ϕ_n the analytic function $\phi(t) = \sum_{i \geqslant 0} a_i t^i$ $(t \to 0)$.

Thus we have $\Delta s_n = n^\nu \lambda^n \phi(1/n)$ and $\Delta s_{n+1} = (n+1)^\nu \lambda^{n+1} \phi_+(1/n)$ with $\phi_+(1/n) = \sum_{j \geqslant 0} b_j/n^j$ and $b_0 = a_0$, $b_1 = a_1$, $b_j = b_j(1)$ $(j \geqslant 2)$ where $b_j(m) = \Delta^{j-1} a_m$.

On the one hand,

$$\lambda_n = \frac{\Delta s_{n+1}}{\Delta s_n} = \lambda\left(1 + \frac{1}{n}\right)^\nu \frac{\phi_+(1/n)}{\phi(1/n)} \quad (a_0 \neq 0).$$

Therefore

$$\lambda(t) = \lambda(1+t)^\nu \frac{\phi_+(t)}{\phi(t)} = \lambda\left(1 + \nu t + \frac{\nu(\nu-1)}{2} t^2 + \cdots\right)\left(1 + \frac{b_1}{a_0} t + \frac{b_2}{a_0} t^2 + \cdots\right)$$

$$\times\left(1 + \frac{a_1}{a_0} t + \frac{a_2}{a_0} t^2 + \cdots\right)^{-1} \quad \text{for } t \to 0.$$

Or

$$\lambda(t) = \lambda + \nu\lambda t + \alpha_2 t^2 + \cdots \quad (t \to 0).$$

Thus

$$\lambda_n = \alpha_0 + \frac{\alpha_1}{n} + \frac{\alpha_2}{n^2} + \frac{\alpha_3}{n^3} + \frac{\alpha_4}{n^4} + \cdots \quad \text{for } n \text{ large enough}$$

with

$$\begin{aligned}
&\alpha_0 = \lambda, \qquad \alpha_1 = \nu\lambda, \qquad \alpha_2 = \lambda\left[\frac{\nu(\nu-1)}{2} - \frac{a_1}{a_0}\right], \\
&\alpha_3 = \lambda\left[\frac{\nu(\nu-1)(\nu-2)}{3!} - (\nu-1)\frac{a_1}{a_0} + \left(\frac{a_1}{a_0}\right)^2 - 2\frac{a_2}{a_0}\right], \\
&\alpha_4 = \lambda\left[\frac{\nu(\nu-1)(\nu-2)(\nu-3)}{4!} - \frac{(\nu-1)(\nu-2)}{2}\frac{a_1}{a_0} - (2\nu-3)\frac{a_2}{a_0} - \frac{3a_3}{a_0}\right. \\
&\qquad \left. + (\nu-1)\left(\frac{a_1}{a_0}\right)^2 + 3\frac{a_1 a_2}{a_0^2} - \left(\frac{a_1}{a_0}\right)^3\right], \ldots .
\end{aligned} \tag{2}$$

On the other hand,

$$\lambda_+\left(\frac{1}{n}\right) = \beta_0 + \frac{\beta_1}{n} + \frac{\beta_2}{n^2} + \cdots \quad (n \to +\infty)$$

with $\beta_0 = \alpha_0$, $\beta_1 = \alpha_1$ and $\beta_j = \beta_j(1) = \Delta^{j-1}\alpha_1$ for $j \geqslant 2$. It follows that

$$\Delta\lambda_n = \lambda_+\left(\frac{1}{n}\right) - \lambda\left(\frac{1}{n}\right) = \frac{-\alpha_1}{n^2} + \frac{-2\alpha_2 + \alpha_1}{n^3} + \frac{-3\alpha_3 + 3\alpha_2 - \alpha_1}{n^4} + \cdots .$$

If k denotes the smallest integer $j \geqslant 1$ such that $\alpha_j \neq 0$, then

$$\Delta\lambda_n = \frac{-k\alpha_k}{n^{k+1}} + \frac{-(k+1)\alpha_{k+1} + \frac{1}{2}k(k+1)\alpha_k}{n^{k+2}} + \cdots = -\frac{k\alpha_k}{n^{k+1}}\left(1 + \frac{\gamma_1}{n} + \cdots\right)$$

$$\text{and} \quad \Delta\lambda_n \sim -\frac{k\alpha_k}{n^{k+1}} \quad (n \to +\infty).$$

Let

$$\psi_n = 1 + \frac{\gamma_1}{n} + \cdots, \qquad \Delta\lambda_n = \frac{-k\alpha_k}{n^{k+1}}\psi_n .$$

Thus

$$\Delta^2\lambda_n = \Delta\lambda_{n+1} - \Delta\lambda_n = \frac{-k\alpha_k}{n^{k+1}}\left[\left(1 + \frac{1}{n}\right)^{-(k+1)}\psi_+(1/n) - \psi(1/n)\right],$$

$$\psi(1/n) = \psi_n \quad \text{and} \quad \psi_+(1/n) = \psi_{n+1} \qquad \text{(Remark 2.5)},$$

$$\Delta^2\lambda_n = -\frac{k\alpha_k}{n^{k+1}}\left(-\frac{(k+1)}{n} + \cdots\right) \quad \text{and} \quad \Delta^2\lambda_n \sim \frac{k(k+1)}{n^{k+2}}\alpha_k \ (n \to +\infty). \qquad \square$$

Among nonlinear transformations, we shall be interested by Aitken's Δ^2 process and Brezinski's θ_2 algorithm, as well as their iterated versions [5,7,15]. We define them as follows. Let us consider a sequence (x_n) such that $\lambda_n = \Delta x_{n+1}/\Delta x_n$ is defined from a certain index N. We suppose $\lambda_n \neq 1$ and $\lambda_n \lambda_{n+1} - 2\lambda_{n+1} + 1 \neq 0$ $\forall n \geqslant N$. (This is true if $(x_n) \in F$.)

If we set

$$R(x_n) = \frac{\Delta x_{n+1}}{1-\lambda_n} \quad \text{and} \quad r(x_n) = \frac{(1-\lambda_n)(1-\lambda_{n+1})}{1-2\lambda_{n+1}+\lambda_n\lambda_{n+1}} R(x_n),$$

then the Δ^2 process applied to (x_n) can be written as

$$x_n^{(1)} = x_{n+1} + R(x_n);$$

and the θ_2 algorithm as

$$\theta(x_n) = x_{n+1} + r(x_n).$$

We have the following result concerning the stability of F_{LIN} by the iterated Δ^2 process.

Lemma 3.2. *Let $(s_n) \in F_{\mathrm{LIN}}$ and let us define the iterated Δ^2 process by*

$$s_n^{(0)} = s_n \quad \text{and} \quad \text{for } p \geqslant 1,\ s_n^{(p)} = s_{n+1}^{(p-1)} + R\left(s_n^{(p-1)}\right).$$

Then $\forall p \geqslant 1$, $(s_n^{(p)}) \in F_{\mathrm{LIN}}$, i.e.

$$\Delta s_n^{(p)} = \lambda^n n^{\nu_p}\left(a_0^{(p)} + \frac{a_1^{(p)}}{n} + \cdots\right) \quad \text{for } n \to +\infty.$$

Proof. It is enough to prove this result for $p = 1$. In fact, we prove that F_{LIN} is stable by application of the Δ^2 process. Let n be sufficiently large. We have

$$\frac{\Delta s_n^{(1)}}{\Delta s_n} = \frac{\lambda_n\, \Delta\lambda_n}{(1-\lambda_n)(1-\lambda_{n+1})}.$$

Since

$$\lambda_n - 1 = (\lambda - 1)\left(1 + \frac{\alpha_k}{(\lambda-1)n^k} + \frac{\alpha_{k+1}}{(\lambda-1)n^{k+1}} + \cdots\right),$$

$$\lambda_{n+1} - 1 = (\lambda - 1)\left(1 + \frac{\beta_k}{(\lambda-1)n^k} + \frac{\beta_{k+1}}{(\lambda-1)n^{k+1}} + \cdots\right),$$

we have

$$\frac{\Delta s_n^{(1)}}{\Delta s_n} \simeq \frac{-\dfrac{k\lambda\alpha_k}{n^{k+1}}\left(1 + \dfrac{\alpha_k}{\lambda n^k} + \cdots\right)\left(1 + \dfrac{(k+1)\alpha_{k+1} - \frac{1}{2}k(k+1)\alpha_k}{k\alpha_k n} + \cdots\right)}{(\lambda-1)^2\left(1 + \dfrac{\alpha_k}{(\lambda-1)n^k} + \cdots\right)\left(1 + \dfrac{\beta_k}{(\lambda-1)n^k} + \cdots\right)}.$$

Thus

$$\Delta s_n^{(1)} = -\Delta s_n \frac{k\lambda\alpha_k}{(\lambda-1)^2 n^{k+1}}\left(1 + \frac{\gamma_1}{n} + \cdots\right). \tag{3}$$

We finally get

$$\Delta s_n^{(1)} = -\frac{k\lambda^{n+1}a_0\alpha_k}{(\lambda-1)^2} n^{\nu-k-1}\left(1 + \frac{\gamma_1'}{n} + \cdots\right),$$

i.e.,

$$\Delta s_n^{(1)} = \lambda^n n^{\nu_1}\left(a_0^{(1)} + \frac{a_1^{(1)}}{n} + \cdots\right) \quad (n \to +\infty)$$

with $\nu_1 = \nu - k - 1$ and $a_0^{(1)} = -k\lambda a_0 \alpha_k/(\lambda-1)^2 \neq 0$, $k \geqslant 1$ being the smallest integer such that $\alpha_k \neq 0$. □

We now give error bounds for the limit of sequences of F_{LIN}, using the iterated Δ^2 process and the θ_2 algorithm. Let $\sigma_n^{(i)} = \theta(s_{n-1}^{(i-1)})$, $i \geqslant 1$, denote the sequence obtained by applying the θ_2 algorithm to the $(i-1)$th column of the iterated Δ^2 process. We have the following theorem.

Theorem 3.3. *If* $(s_n) \in F_{\mathrm{LIN}}$, *then* $\exists n_0 \geqslant 1$, $\forall n \geqslant n_0$:

$$s^* \in \left[\min\left(s_n^{(i)}, \sigma_n^{(i)}\right), \max\left(s_n^{(i)}, \sigma_n^{(i)}\right)\right] \quad \forall i \geqslant 1.$$

Proof. Let us consider only the case $i = 1$ (Lemma 3.2). Let $k \geqslant 1$ be the smallest integer such that $\alpha_k \neq 0$ and let n be large enough.

We have $\Delta\sigma_n^{(1)}/\Delta s_n = \lambda_n \phi_{n-1}/D_n D_{n-1}$ where $\phi_n = (1-\lambda_n)\,\Delta^2\lambda_n + \Delta\lambda_n\,\Delta\lambda_{n+1} + (\Delta\lambda_n)^2$ and $D_n = 1 - 2\lambda_{n+1} + \lambda_n\lambda_{n+1}$.

Since $\lim_{n\to+\infty}\lambda_n = \lambda$, $\lim_{n\to+\infty} n^{k+2}\,\Delta^2\lambda_n = k(k+1)\alpha_k$ and $\lim_{n\to+\infty} n^{k+1}\,\Delta\lambda_n = -k\alpha_k$ (Lemma 3.1), we have $\lim_{n\to+\infty} D_n = (1-\lambda)^2$ and $\lim_{n\to+\infty} n^{k+2}\phi_n = k(k+1)\alpha_k(1-\lambda)$. Thus

$$\frac{\Delta\sigma_n^{(1)}}{\Delta s_n} \sim \frac{k(k+1)\lambda\alpha_k}{(1-\lambda)^3 n^{k+2}} = \frac{M}{n^{k+2}} \quad (M \neq 0) \qquad (n \to +\infty).$$

On the other hand, from the A.E. (3), we have

$$\frac{\Delta s_n^{(1)}}{\Delta s_n} \sim \frac{-k\lambda\alpha_k}{(1-\lambda)^2 n^{k+1}} = \frac{L}{n^{k+1}} \quad (L \neq 0) \qquad (n \to +\infty).$$

Since $M/L = -(k+1)/(1-\lambda) < 0$, we deduce that $(\Delta\sigma_n^{(1)}/\Delta s_n)\cdot(\Delta s_n^{(1)}/\Delta s_n) < 0$, namely: $\Delta\sigma_n^{(1)}\cdot\Delta s_n^{(1)} < 0$ from a certain index N. Hence, from Lemma 2.4, we see that $s^* - \sigma_n^{(1)}$ and $s^* - s_n^{(1)}$ have opposite signs, for $n \geqslant N$. □

Thus we see that, for a given sequence of F_{LIN}, the simultaneous application of the Δ^2 and θ_2 processes provides an interval asymptotically containing the limit. Further one may iterate the method several times to get a good approximation of the limit.

Let us now study the case of F_{LOG}.

4. Error bounds for $F_{\mathrm{LOG}}^{(\nu)}$

In this section, we deal with the family $F_{\mathrm{LOG}}^{(\nu)}$, i.e., with the sequences such that

$$\Delta s_n = n^{\nu}\left(a_0 + \frac{a_1}{n} + \frac{a_2}{n^2} + \cdots\right), \quad \text{for } n \to \infty,$$

with $a_0 \neq 0$ and $\nu < -1$.

4.1. Modified Aitken Δ^2 process

We know that the θ_2 algorithm accelerates a large class of logarithmic sequences, in particular F_{LOG} [5,7,15].

We now determine another transformation which accelerates F_{LOG}, and which, with the θ_2 process provides an estimation of the error.

For this purpose, we will study a modified version of Aitken's process which generalizes that given by Sablonnière [12]. It consists in

$$t_n^{(1)} = s_{n+1} + \rho R(s_n),$$

where ρ is a real number, which will be determined later for the sequences of the $F_{\text{LOG}}^{(\nu)}$.

Before that, a more precise study of the sequence (λ_n) is needed.

Lemma 4.1. *If $(s_n) \in F_{\text{LOG}}^{(\nu)}$, then, for n large enough*:

$$\lambda_n = 1 + \frac{\alpha_1}{n} + \frac{\alpha_2}{n^2} + \cdots, \quad \textit{with } \alpha_1 = \nu,\ \alpha_2 = \frac{\nu(\nu-1)}{2} - \frac{a_1}{a_0}, \ldots .$$

Further, when $\alpha_2 \neq 0$, this expansion can also be written as

$$\lambda_n = 1 + \frac{\alpha_1}{m} + \frac{\delta_k}{m^k} + \frac{\tilde{\delta}_{k+1}}{m^{k+1}} + \cdots,$$

with $\delta_j = \alpha_j - \alpha^{j-2}\alpha_2$, $\tilde{\delta}_j = \tilde{\delta}_j(1)$ where $\tilde{\delta}_j(p) = \Delta^{j-1}(\alpha^{-p}\alpha_p) = \Delta^{j-1}(\alpha^{-p}\delta_p)$, $m = n - \alpha$, $\alpha = \alpha_2/\alpha_1$ and $k \geqslant 3$ being the smallest integer such that $\delta_k \neq 0$. In particular $\tilde{\delta}_{k+1} = \delta_{k+1} - k\alpha\delta_k$.

Proof. The first expansion of λ_n is obtained in the same way as in Lemma 3.1, with $\lambda = 1$.

Now, if $\alpha_2 \neq 0$, let us set $\alpha = \alpha_2/\alpha_1 \neq 0$ and $\lambda(x) = 1 + \alpha_1 x + \alpha_2 x^2 + \cdots$ $(x \to 0)$. Let y be a small enough real number such that $1 + \alpha y \neq 0$. Let $x = y/(1 + \alpha y)$. Thus we have

$$\lambda(x) = 1 + \alpha_1 y(1+\alpha y)^{-1} + \alpha_2 y^2 (1 + \alpha y)^{-2} + \cdots.$$

Developing, we get

$$\lambda(x) = 1 + \alpha_1 y + \tilde{\delta}_3 y^3 + \tilde{\delta}_4 y^4 + \cdots + \tilde{\delta}_j y^j + \cdots \quad (x \to 0),$$

where

$$\tilde{\delta}_j = \alpha^j \sum_{i=0}^{j-1} (-1)^{j-i-1} \binom{j-1}{i} \alpha^{-(i+1)} \alpha_{i+1}.$$

If we set $\tilde{\delta}_j(p) = \alpha^j\, \Delta^{j-1}(\alpha^{-p}\alpha_p)$ $(j \geqslant 3)$, then $\tilde{\delta}_j(1) = \tilde{\delta}_j$. In particular, as $\delta_j = \alpha_j - \alpha^{j-2}\alpha_2$ and $\alpha^2\alpha_1 = \alpha\alpha_2$, then $\tilde{\delta}_3 = \alpha^3(\alpha^{-3}\alpha_3 - 2\alpha^{-2}\alpha_2 + \alpha^{-1}\alpha_1) = \alpha_3 - \alpha\alpha_2 = \delta_3$ and $\tilde{\delta}_4 = \delta_4 - 3\alpha\delta_3$.

More generally, we show by induction that

$$\tilde{\delta}_j(p) = \alpha^j\, \Delta^{j-1}(\alpha^{-p}\delta_p) \quad (j \geqslant 3).$$

It follows that if $\delta_3 = \cdots = \delta_{k-1} = 0$ and $\delta_k \neq 0$, then $\tilde{\delta}_3(1) = \cdots = \tilde{\delta}_{k-1}(1) = 0$, $\tilde{\delta}_k(1) = \delta_k$ and $\tilde{\delta}_{k+1}(1) = \delta_{k+1} - k\alpha\delta_k$. Consequently, if we take $x = 1/n$ and $m = n - \alpha$, then $y = 1/m$ and

$$\lambda_n = 1 + \frac{\nu}{m} + \frac{\delta_k}{m^k} + \frac{\delta_{k+1} - k\alpha\delta_k}{m^{k+1}} + \cdots \quad (n \to +\infty). \qquad \square$$

Remark 4.2. The coefficients δ_j were introduced by Lubkin [11]. They characterize the kernel of his transformation, which differs from that of the θ_2 algorithm only by a shift in the indexes.

Actually, if a sequence (s_n) is such that $\delta_j = 0$ $\forall j \geqslant 3$, then $\lambda_n = 1 + \nu/(n-\alpha)$ and $\sigma_n^{(1)} = s^*$ $(n \geqslant N)$.

We will now determine the real number ρ which enters into the modified version of the Δ^2 process in order to accelerate $F_{\text{LOG}}^{(\nu)}$.

Lemma 4.3. *Let $(s_n) \in F_{\text{LOG}}^{(\nu)}$. A necessary and sufficient condition for $t^{(1)} : (s_n) \to (t_n^{(1)})$ to accelerate $F_{\text{LOG}}^{(\nu)}$ is that $\rho = \nu/(\nu+1)$ $(\nu < -1)$.*

Moreover, we have $(t_n^{(1)}) \in F_{\text{LOG}}^{(\nu_1)}$ where $\nu_1 = \nu - k + 1$ and

$$n^{k-2}\left(t_n^{(1)} - s^*\right) = o\left(s_n - s^*\right);$$

$k \geqslant 3$ being the smallest integer such that $\delta_k \neq 0$.

If $\delta_j = 0$ $\forall j \geqslant 3$, then we have $t_n^{(1)} = s^*$ $(n \geqslant N)$.

Proof. We have

$$\frac{\Delta t_n^{(1)}}{\Delta s_{n+1}} = 1 + \rho\mu_n \quad \text{with } \mu_n = \frac{(\lambda_{n+1} - 1) - \lambda_{n+1}(\lambda_n - 1)}{(\lambda_n - 1)(\lambda_{n+1} - 1)}.$$

Using the same notations as in Lemma 4.1, we have

$$\lambda_n = 1 + \frac{\nu}{m} + \frac{\delta_k}{m^k} + \frac{\tilde{\delta}_{k+1}}{m^{k+1}} + \cdots \quad (n \to \infty),$$

$$\lambda(x) = 1 + \nu y + \delta_k y^k + \tilde{\delta}_{k+1} y^{k+1} + \cdots \quad (x \to 0).$$

Let us set

$$D(y) = D = \nu + \delta_k y^{k-1} + \tilde{\delta}_{k+1} y^k + \cdots.$$

Thus

$$D_+(y) = D_+ = \nu + \delta_k y^{k-1} + \left(\tilde{\delta}_{k+1} - (k-1)\delta_k\right) y^k + \cdots.$$

It follows that

$$\lambda(x) = 1 + yD(y), \qquad \lambda_+(x) = 1 + \frac{y}{1+y} D_+(y)$$

and

$$\mu(x) = \frac{D_+ - (1+y)\cdot D - y \cdot D \cdot D_+}{y \cdot D \cdot D_+} = \frac{-\nu(1+\nu)y - (2\nu + k)\delta_k y^k + \cdots}{\nu^2 y + 2\nu\delta_k y^k + \cdots}.$$

In other words

$$\mu(x) = -\frac{\nu+1}{\nu}\left(1 + \frac{(k-2)}{\nu(\nu+1)}\delta_k y^{k-1} + \cdots\right).$$

Therefore

$$\frac{\Delta t_n^{(1)}}{\Delta s_{n+1}} = 1 - \rho\frac{\nu+1}{\nu}\left(1 + \frac{(k-2)}{\nu(\nu+1)}\frac{\delta_k}{m^{k-1}} + \cdots\right).$$

Thus if $\rho = \nu/(\nu+1)$, then

$$\frac{\Delta t_n^{(1)}}{\Delta s_{n+1}} = -\frac{(k-2)\delta_k}{\nu(\nu+1)m^{k-1}} + \cdots$$

and

$$\frac{\Delta t_n^{(1)}}{\Delta s_{n+1}} \sim -\frac{(k-2)\delta_k}{\nu(\nu+1)n^{k-1}} \quad \text{for } n \to \infty. \tag{4}$$

As $\Delta s_{n+1} = n^\nu(a_0 + (a_1 + \nu a_0)/n + \cdots)$, we have

$$\Delta t_n^{(1)} = -\frac{(k-2)\delta_k}{\nu(\nu+1)n^{k-1}}\left(1 + \frac{A}{n} + \cdots\right)n^\nu\left(a_0 + \frac{B}{n} + \cdots\right).$$

So $\Delta t_n^{(1)} = n^{\nu_1}(a_0^{(1)} + a_1^{(1)}/n + \cdots)$ $(n \to \infty)$, with $\nu_1 = \nu - k + 1 < -1$ since $\nu < -1$, $k > 2$, and $a_0^{(1)} = -(k-2)\delta_k a_0/\nu(\nu+1) \neq 0$. It follows that $(t_n^{(1)}) \in F_{\mathrm{LOG}}^{(\nu_1)}$.

On the other hand

$$s^* - s_{n+1} \sim -\frac{n^{\nu+1}}{(\nu+1)}a_0 \qquad (n \to \infty),$$

$$s^* - t_{n+1}^{(1)} \sim -\frac{n^{\nu-k+2}}{(\nu-k+2)}a_0^{(1)} \quad (n \to \infty),$$

according to Lemma 2.3. Consequently $\lim_{n \to +\infty} n^{k-2}(t_n^{(1)} - s^*)/(s_n - s^*) = 0$ and, in particular $t^{(1)}$ accelerates $F_{\mathrm{LOG}}^{(\nu)}$.

Finally, if $\delta_j = 0$ $\forall j \geqslant 3$, then $\lambda_n = 1 + \nu/(n-\alpha)$ and $\mu_n = -(\nu+1)/\nu$. Thus $\Delta t_n^{(1)} = 0$ and $t_n^{(1)} = s^*$ for $n \geqslant N$, since it is a convergent sequence. □

Remarks 4.4. (i) As mentioned by Benchiboun [3] in his discussion of the modified B process, another interpretation of the θ_2 algorithm can be given based on Aitken's process.

The constant $\rho = \nu/(\nu+1)$ in the modified Δ^2 process is replaced by a sequence (ρ_n) which converges to ρ.

We then have $\sigma_{n+1}^{(1)} = \theta(s_n) = s_{n+1} + \rho_n\ R(s_n)$, where

$$\rho_n = \frac{(1-\lambda_n)(1-\lambda_{n+1})}{(1-\lambda_{n+1}) - \lambda_{n+1}(1-\lambda_n)} = -\mu_n^{-1}.$$

Thus

$$\rho_n = \frac{\nu}{\nu+1}\left(1 - \frac{(k-2)\delta_k}{\nu(\nu+1)n^{k-1}} + \cdots\right) \quad (n \to \infty)$$

and $\lim_{n \to +\infty}\rho_n = \nu/(\nu+1)$. Hence ν can be estimated by the sequence

$$\nu_n(\theta) = -\frac{(1-\lambda_n)(1-\lambda_{n+1})}{\Delta\lambda_n}. \tag{5}$$

(ii) An important property due to Lubkin [11], which we will use later, is the following. If

$$\lambda_n = 1 + \frac{\nu}{m} + \frac{\delta_k}{m^k} + \cdots \quad (n \to \infty),$$

then

$$\frac{\Delta\sigma_n^{(1)}}{\Delta s_{n+1}} \sim -\frac{(k-1)(k-2)\delta_k}{\nu(\nu+1)^2 n^{k-1}} \quad (n \to \infty). \tag{6}$$

We now give a result on the estimation of the limit of sequences of $F_{\text{LOG}}^{(\nu)}$.

Theorem 4.5. *If* $(s_n) \in F_{\text{LOG}}^{(\nu)}$, *then* $\exists N \geqslant 1$, $\forall n \geqslant N$:

$$s^* \in \left[\min\left(t_n^{(1)}, \sigma_n^{(1)}\right), \max\left(t_n^{(1)}, \sigma_n^{(1)}\right)\right].$$

Proof. Let us assume that $k \geqslant 3$ is the smallest integer such that $\delta_k \neq 0$ (if $\delta_j = 0$ $\forall j \geqslant 3$, then $t_n^{(1)} = \sigma_n^{(1)} = s^*$ for n large enough). Using (4) and (6), we have

$$\frac{\Delta\sigma_n^{(1)}}{\Delta s_{n+1}} \sim \frac{M'}{n^{k-1}} \quad (n \to \infty) \qquad \text{with } M' = -\frac{(k-1)(k-2)\delta_k}{\nu(\nu+1)^2}$$

and

$$\frac{\Delta t_n^{(1)}}{\Delta s_{n+1}} \sim \frac{L'}{n^{k-1}} \quad (n \to \infty) \qquad \text{with } L' = -\frac{(k-2)\delta_k}{\nu(\nu+1)}.$$

As $M'/L' = (k-1)/(\nu+1) < 0$ $(k \geqslant 3,\ \nu < -1)$, then $\Delta\sigma_n^{(1)} \cdot \Delta t_n^{(1)} < 0$ and $(s^* - \sigma_n^{(1)}) \cdot (s^* - t_n^{(1)}) < 0$ (Lemma 2.4) from a certain index. □

Thus we have solved the problem of the control of the error for sequences of $F_{\text{LOG}}^{(\nu)}$. In fact, given a sequence transformation, we have constructed a second one and an interval containing asymptotically the limit. In order to improve this estimation we have to check, as for sequences of F_{LIN}, the possibility of iterating the method several times.

Let us recall that the transformation $t^{(1)}$ is given by $t_n^{(1)} = s_{n+1} + (\nu/(\nu+1))R(s_n)$, $(s_n) \in F_{\text{LOG}}^{(\nu)}$. We saw that $F_{\text{LOG}}^{(\nu_1)} \supset t^{(1)}(F_{\text{LOG}}^{(\nu)})$, where $\nu_1 = \nu - k + 1$ $(k \geqslant 3)$.

4.2. Generalization of the iterated modified Δ^2 process

Since $(t_n^{(1)})$ is a sequence of $F_{\text{LOG}}^{(\nu_1)}$, our iterated version of the modified Δ^2 process consists in applying to $(t_n^{(1)})$, the transformation $t^{(2)}$ in order to obtain the sequence $(t_n^{(2)})$

$$t_n^{(2)} = t_{n+1}^{(1)} + \frac{\nu_1}{\nu_1+1} R\left(t_n^{(1)}\right).$$

This is a sequence of $F_{\text{LOG}}^{(\nu_2)}$, which converges faster than $(t_n^{(1)})$ (Lemma 4.3), and which, with $\sigma_n^{(2)} = \theta(t_{n-1}^{(1)})$, controls the error (Theorem 4.5). In general, $(t_n^{(p-1)})$ and ν_{p-1} being determined, we define $(t_n^{(p)})$ by

$$t_n^{(p)} = t^{(p)}\left(t_n^{(p-1)}\right) = t_{n+1}^{(p-1)} + \frac{\nu_{p-1}}{\nu_{p-1}+1} R\left(t_n^{(p-1)}\right), \tag{7}$$

$$n \geqslant 1, \quad p \geqslant 1, \quad \text{with } t_n^{(0)} = s_n, \quad \nu_0 = \nu, \quad k_0 = k.$$

So we have, by induction, the following result.

Theorem 4.6. *If* $(s_n) \in F_{\mathrm{LOG}}^{(\nu)}$, *then*

(i) $\forall p \geqslant 1$, $(t_n^{(p)}) \in F_{\mathrm{LOG}}^{(\nu_p)}$, *with* $\nu_p = \nu_{p-1} - k_{p-1} + 1$ *and* $n^{k_{p-1}-2}(t_n^{(p)} - s^*) = o(t_n^{(p-1)} - s^*)$, $j = k_{p-1}$ *being the smallest integer such that*

$$\delta_j^{(p-1)} = \alpha_j^{(p-1)} - \left(\frac{\alpha_2^{(p-1)}}{\alpha_1^{(p-1)}}\right)^{j-2} \alpha_2^{(p-1)} \neq 0$$

where

$$\left(\alpha_i^{(p-1)}\right):\ \lambda_n^{(p-1)} = \frac{\Delta t_{n+1}^{(p-1)}}{\Delta t_n^{(p-1)}} = 1 + \frac{\alpha_1^{(p-1)}}{n} + \frac{\alpha_2^{(p-1)}}{n^2} + \cdots \quad \textit{for } n \to +\infty;$$

$\alpha_1^{(p-1)} = \nu_{p-1} < -1.$

(ii) $\exists N' \geqslant 1$, $\forall n \geqslant N'$, $\forall p \geqslant 1$:

$$s^* \in \left[\min\left(t_n^{(p)}, \sigma_n^{(p)}\right), \max\left(t_n^{(p)}, \sigma_n^{(p)}\right)\right],$$

where

$$\sigma_n^{(p)} = \theta\left(t_{n-1}^{(p-1)}\right).$$

Remarks 4.7. (i) The knowledge of ν_p, that is of the integer $k_{p-1} \geqslant 3$, characterizes the transformation $t^{(p)}$. We have

$$\nu_p = \nu - \sum_{j=0}^{p-1} k_j + p.$$

In particular, if all the integers k_j $(j = 0, \ldots, p-1)$ are equal to 3, then

$$\nu_p = \nu - 2p. \tag{8}$$

(ii) This iterated version of the modified Δ^2 process (7) generalizes the method proposed by Sablonnière [12] for fixed point sequences of F_{LOG}. In [12] the modified Δ^2 process is

$$s_n^{(p)} = s_{n+1}^{(p-1)} + \frac{2p}{2p-1} R\left(s_n^{(p-1)}\right) \quad \text{if } (s_n) \in F_{\mathrm{LOG}}^{(-2)}$$

and

$$s_n^{(p)} = s_{n+1}^{(p-1)} + \frac{4p-1}{4p-3} R\left(s_n^{(p-1)}\right) \quad \text{if } (s_n) \in F_{\mathrm{LOG}}^{(-3/2)}.$$

If $p = 1$, we have, in the first case

$$\rho_{p-1} = \frac{2p}{2p-1} = 2 = \frac{\nu}{\nu+1} \quad \text{since } \nu = -2,$$

and, in the second case

$$\rho_{p-1} = \frac{4p-1}{4p-3} = 3 = \frac{\nu}{\nu+1} \quad \text{since } \nu = -\tfrac{3}{2}.$$

For $p \geqslant 2$, the choices of $\rho_{p-1} = 2p/(2p-1)$ when $\nu = -2$ and $\rho_{p-1} = (4p-1)/(4p-3)$ when $\nu = -\frac{3}{2}$ are no more valid in general.

In fact, they correspond to our choice of ρ_{p-1} in the particular case where $k_j=3$ for $j=0,1,\ldots,p-2$. By using (8),

$$\rho_{p-1}=\frac{\nu_{p-1}}{\nu_{p-1}+1}=\frac{\nu-2p+2}{\nu-2p+3}=\begin{cases}\dfrac{2p}{2p-1} & \text{if } \nu=-2,\\[2mm] \dfrac{4p-1}{4p-3} & \text{if } \nu=-\frac{3}{2}.\end{cases}$$

To illustrate this situation, let us consider the following example:

$$(s_n)\quad \text{where } s_n=\sum_{j=1}^{n-1}\left(\frac{1}{4j^2-1}+\frac{1}{j^5}\right).$$

$(s_n)\in F_{\mathrm{LOG}}^{(-2)}$ since

$$\Delta s_n=n^{-2}\left(\frac{1}{4}+\frac{1}{16n^2}+\frac{1}{n^3}+\frac{1}{64n^4}+\cdots\right)\quad \text{for } n\to+\infty.$$

According to the previous notations, we have

$$\begin{cases}a_1=0,\ a_3=1 \text{ and } a_{2j+1}=0 & \forall j\geqslant 2,\\ a_{2j}=\left(\frac{1}{2}\right)^{2j+2} & \forall j\geqslant 0.\end{cases}$$

We deduce from formulas (2) ($\lambda=1$), that

$$\alpha_1=\nu=-2,\quad \alpha_2=3,\quad \alpha_3=-\tfrac{9}{2}\quad \text{and}\quad \alpha_4=-\tfrac{21}{4}.$$

We get $\delta_3=0$ and $\delta_4\neq 0$ and hence $k_0=4$.

As we noticed, our repeated version of the modified Δ^2 process needs, at each step $(p+1)$, the knowledge of ν_p. According to Theorem 4.6 we have

$$\nu_p=\nu_{p-1}-k_{p-1}+1, \tag{9}$$

where k_{p-1} is an integer $\geqslant 3$ determined by the coefficients $\delta_j^{(p)}$. But in numerical applications, it is often impossible to know whether a given $\delta_j^{(p)}$ is zero or not.

However, this problem can be solved by giving an estimation of ν_p from $(t_n^{(p)})$.

It will be useful only for controlling the integer k_{p-1}, which is used to obtain ν_p by (9).

For this we can proceed in two ways. The simplest one is based on formula (5) in Remark 4.4. The other one, using the same principle as Edmund and Gordon [10] for estimating the convergence orders in iterated Richardson extrapolation, requires the resolution of a nonlinear equation.

Thus we find an estimation $\nu_p(n)$ of ν_p by solving the following equation

$$\phi(x)=\left(1+\frac{2}{n}\right)^{x+1}-\left(\lambda_n^{(p)}+1\right)\left(1+\frac{1}{n}\right)^{x+1}+\lambda_n^{(p)}. \tag{10}$$

It is easy to see that ϕ has two zeros, one of them being $x=-1$. The secant method is used to obtain the second one.

5. Numerics

Let us give numerical examples concerning the error bounds for sequences of F, by using two transformations. In particular, the estimation of the convergence orders in the repeated modified Δ^2 process will be illustrated. The final results are summarized in Tables 1–5. We consider only the last interval in each column.

In the logarithmic case, we add the exact value of ν. In the last columns of the tables, the estimations $\nu^{(1)}$ and $\nu^{(2)}$ appear, the first one being the approximate solution of equation (10) and the second computed from formula (5).

5.1. Linear case

We compute the sums of the two series

$$\sum_{k\geqslant 0} \frac{(0.8)^k}{k+1}, \qquad \sum_{k\geqslant 0} \frac{(-1)^k}{2k+1}. \tag{11, 12}$$

Table 1
Error bounds for the linear monotone sequence (11)

k	n	$s^* - s_n^{(k)}$	$s^* - \sigma_n^{(k)}$
1	22	$0.22\cdot 10^{-4}$	$-0.13\cdot 10^{-4}$
2	19	$0.17\cdot 10^{-5}$	$-0.76\cdot 10^{-6}$
3	14	$0.53\cdot 10^{-6}$	$-0.22\cdot 10^{-6}$

Table 2
Error bounds for the linear oscillatory sequence (12)

k	n	$s^* - s_n^{(k)}$	$s^* - \sigma_n^{(k)}$
1	17	$-0.11\cdot 10^{-4}$	$0.29\cdot 10^{-4}$
2	13	$-0.60\cdot 10^{-7}$	$0.15\cdot 10^{-6}$
3	11	$-0.52\cdot 10^{-9}$	$0.11\cdot 10^{-8}$
4	8	$0.13\cdot 10^{-10}$	$-0.34\cdot 10^{-10}$
5	5	$-0.27\cdot 10^{-11}$	0.000...

Table 3
Sequence (13), $s(18) = 1.59\ldots$, $s^* = 1.64493406684\ldots$

k	n	ν_{k-1}	$t_n^{(k)}$	$\sigma_n^{(k)}$	$\nu^{(1)}$	$\nu^{(2)}$
1	12	-2	1.64498	1.64486	-3.85	-4.03
2	9	-4	1.6449339	1.6449341	-5.19	-6.15
3	6	-6	1.6449340692	1.6449340662	-5.81	-8.64
4	3	-8	1.64493406665	1.64493406688	*	*

Table 4
Sequence (14), $s(18) = 0.993\ldots$, $s^* = 1.71379673481\ldots$

k	n	ν_{k-1}	$t_n^{(k)}$	$\sigma_n^{(k)}$	$\nu^{(1)}$	$\nu^{(2)}$
1	12	$-\sqrt{2}$	1.719	1.690	-3.38	-3.39
2	9	$-2-\sqrt{2}$	1.713812	1.713785	-5.11	-5.65
3	6	$-4-\sqrt{2}$	1.7137963	1.7137968	-5.43	-7.55
4	3	$-6-\sqrt{2}$	1.7137967498	1.7137967314	*	*

Table 5
Sequence (15), $s(17) = 1.50\ldots$, $s^* = 1.517343061991\ldots$

k	n	ν_{k-1}	$t_n^{(k)}$	$\sigma_n^{(k)}$	$\nu^{(1)}$	$\nu^{(2)}$
1	12	-2	1.5173391	1.5173560	-5.76	-6.11
2	9	-6	1.517343098	1.517343051	-7.16	-8.12
3	6	-8	1.51734306144	1.51734306212	*	*

5.2. *Logarithmic case*

We compute the sums of the three series

$$\sum_{k\geqslant 1} \frac{1}{k^2}, \qquad \sum_{k\geqslant 1} \left(k + \frac{e^{(-1/k)}}{k}\right)^{-\sqrt{2}}, \qquad \sum_{k\geqslant 1} \left(\frac{1}{4k^2-1} + \frac{1}{k^6}\right). \tag{13, 14, 15}$$

We remark that the estimations $\nu^{(1)}$ and $\nu^{(2)}$, with the theoretical result (9), allow us to confirm the exact value of ν. Both methods give comparable results. But ours is simpler because it requires an immediate calculation from the terms of sequence.

Finally, let us remark that, in the case, the intervals given by the control process contain the limit from $n = 1$ and not only asymptotically.

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